Ricci curvature, minimal surfaces and sphere theorems

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Abstract

Using an analogue of Myers' theorem for minimal surfaces and three dimensional topology, we prove the diameter sphere theorem for Ricci curvature in dimension three and a corresponding eigenvalue pinching theorem. This settles these two problems for closed manifolds with positive Ricci curvature since they are both false in dimensions greater than three.

§1. Introduction

In this paper, we consider n-dimensional Riemannian manifolds with Ricci curvature $Ric \geq n - 1$. Recall that under this condition, Myers' theorem implies that the diameter diam $\leq \pi$. Cheng [Ch], using an eigenvalue comparison method, showed that equality is achieved in Myers' theorem, i.e., diam $= \pi$, if and only if the manifold is isometric to the standard sphere. This improved an earlier result of Toponogov [To] who obtained the same conclusion under the condition that the sectional curvature sec ≥ 1.

Associated to Toponogov's metric rigidity theorem, there is a topological stability theorem due to Grove-Shiohama [GS] that says if sec ≥ 1 and diam $> \pi/2$ then M is homeomorphic to a sphere. Efforts were made to

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establish a similar topological stability result for Ricci curvature associated to Cheng's rigidity theorem, see [Es], [GP], [Sh], [PZ] and [Pe], among others. All these results assume in addition some conditions on the sectional curvature. This turned out to be necessary, since Anderson [An] and Otsu [Ot] constructed examples in dimension ≥ 4 that satisfy $Ric \geq n-1, \; \text{diam} \geq \pi - \epsilon$ for any ϵ, but are not spheres. Thus the only dimension left is dimension three. Using the result in [Zh], Wu [Wu] proved the three dimensional sphere theorem under an additional condition that the volume $\text{vol} > v > 0$. Our first theorem is to get rid of this volume condition thus settles the problem for Ricci curvature:

Theorem 1 *There is a constant $\epsilon > 0$ ($\approx 0.47\pi$), such that if M^3 is a three dimensional manifold satisfying:*

$$Ric \geq 2, \quad diam > \pi - \epsilon,$$

then M^3 is diffeomorphic to S^3.

We conjecture that the result is true for $\epsilon = \pi/2$.

There are similar results about the rigidity and stability for the first eigenvalues. Recall that Lichnerowicz [Li] proved that $\lambda_1 \geq n$ if $Ric \geq n-1$, and when $\lambda_1 = n$, Obata [Ob] proved M is isometric to the standard sphere. Many authors tried to prove a stabilty result that says if $\lambda < n + \epsilon$, then it is a sphere. Under the stronger condition that $sec \geq 1$, Li-Zhong[LZ], Li-Treibergs [LT] proved a sphere theorem in lower dimensions, the problem was eventually solved by Croke [Cr]. The similar question for Ricci curvature again turns out to be false in dimension ≥ 4 [An]. It is a direct consequence of our theorem 1 and a result of Croke (Theorem B in [Cr]) that the stability theorem is true for Ricci curvature in dimension three, namely:

Theorem 2 *There is a constant $\delta > 0$ (≈ 1.91), such that if M^3 is a three dimensional manifold satisfying:*

$$Ric \geq 2, \quad \lambda_1 < 3 + \delta,$$

then M is diffeomorphic to S^3.

§2. The proof

Our idea depends in essential ways on works of Hamilton [Ha] and Schoen-Yau [SY1]. We adopt part of the proof to our situation with the help of three dimensional topology. This part was analogous to the argument in [Zh]. Another main ingredient is a Myers' theorem for minimal surfaces essentially contained in [SY2].

Lemma 1 *For any $\epsilon > 0$, there is a positive number $\tau(\epsilon)$ with $\lim_{\epsilon \to 0} \tau = 0$ such that if M^n is a n-dimensional Riemannian manifold satisfying*

$$Ric \geq n - 1, \quad diam \geq \pi - \epsilon,$$

then every element of $\pi_1(M, p)$ can be represented by a geodesic loop at p of length $\leq \tau$. Here p is a point that realizes the diameter of M.

Proof Let p, q be two points in M with $\text{dist}(p, q) = \text{diam}$. Let $\pi : \tilde{M} \to M$ be the unversal covering map and $\tilde{p} \in \pi^{-1}(p), \tilde{q} \in \pi^{-1}(q)$. For any $[\sigma] \in \pi_1(M, p)$ with σ a minimal geodesic loop at p, let $\tilde{\sigma}$ be a lifting of σ with base point $\tilde{p}$, then $\tilde{\sigma}(1) \in \pi^{-1}(p)$, and therefore

$$\text{dist}(\tilde{\sigma}(1), \tilde{q}) \geq \pi - \epsilon, \quad \text{dist}(\tilde{p}, \tilde{q}) \geq \pi - \epsilon,$$

Let $\tau = \text{dist}(\tilde{p}, \tilde{\sigma}(1))$. Since the balls $B_{\tilde{p}}(\tau/2)$ and $B_{\tilde{\sigma}(1)}(\tau/2)$ are disjoint, one concludes

$$\text{vol}(M) \geq \text{vol}(B_{\tilde{p}}(\tau/2)) + \text{vol}(B_{\tilde{\sigma}(1)}(\tau/2)) + \text{vol}(B_{\tilde{q}}(\pi - \epsilon - \tau/2)).$$

By the relative volume comparison theorem,

$$
\begin{aligned}
1 &\geq \frac{\text{vol}(B_{\tilde{p}}(\tau/2))}{\text{vol}(\tilde{M})} + \frac{\text{vol}(B_{\tilde{\sigma}(1)}(\tau/2))}{\text{vol}(\tilde{M})} + \frac{\text{vol}(B_{\tilde{q}}(\pi - \epsilon - \tau/2))}{\text{vol}(\tilde{M})} \\
&\geq \frac{\text{vol}(B^1(\tau/2))}{\text{vol}(S^n)} + \frac{\text{vol}(B^1(\tau/2))}{\text{vol}(S^n)} + \frac{\text{vol}(B^1(\pi - \epsilon - \tau/2))}{\text{vol}(S^n)}
\end{aligned}
$$

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where $B^1(r)$ is a ball of radius r in the standard sphere S^n. It follows that $\lim_{\epsilon \to 0} \tau = 0$. q.e.d.

For a simple closed curve $\Gamma \subset M$ that bounds a disk, we define

$Rad(\Gamma) = \sup\{r : \ \Gamma$ does not bound a stable minimal disk in the r neighborhood of $\Gamma\}$.

Lemma 2 *If M^3 is a three dimensional manifold with Ricci curvature $Ric \geq 2$, then we have*

$$Rad(\Gamma) \leq \frac{4\sqrt{2}}{9}\pi.$$

Proof This type of estimates was first introduced by Scheon-Yau [SY2] in the context of the size of black-holes by assuming a positive lower bound on the scalar curvature. In the presence of a positive lower bound on the Ricci curvature, we can improve their estimate for $Rad(\Gamma)$, using a different test function. For completeness, we present a detailed proof in this case.

We denote the stable minimal surface under consideration by Σ (In our application, we will actually have an area minimizing surface). By the second variation formula, one has

$$\int_{\Sigma} (|\nabla \psi|^2 - \psi^2 |h|^2 - \psi^2 Ric(\nu)) \geq 0$$

for all ψ vanishing at Γ, where $Ric(\nu)$ is the Ricci curvature of the ambient manifold in the direction ν which is the unit normal vector to Σ and h is the second fundamental form. If we shrink the surface Σ a little bit, we know, by Theorem 1 in [FS], there is a positive function g on the new minimal surface which is again denoted by Σ, such that

$$\triangle g + |h|^2 g + Ric(\nu)g = 0. \tag{1}$$

Let ρ be any number that is $< Rad(\Gamma)$. Fix a point x in Σ with $dist(x, \Gamma) \geq \rho$. Let γ be any curve on Σ from x to Γ and f be a positive function on Σ. We consider the functional

$$I[\gamma] = \int_{\gamma} f(\gamma(s))ds$$

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where the f is a function of g which will be chosen later.

We minimize this functional over all such curves γ (thus $f = 1$ in the usual variational proof of Myers theorem). Let γ be the minimizing curve with length $l \geq \rho$. The nonnegativity of the second variation implies that

$$\int_0^l \{\varphi^2 \triangle_\Sigma f - \varphi^2 \frac{d^2 f}{dt^2} + f\dot{\varphi}^2 - \varphi^2 f^{-1} |(\nabla f)^\perp|^2\}$$
$$\geq \int_0^l \{f\varphi^2 R(e, \dot{c}, e, \dot{c}) + f\varphi^2 h(\dot{c}, \dot{c})h(e, e)\}dt.$$

where $\{e, \dot{c}\}$ is an orthonormal basis for the surface Σ, $R(e, \dot{c}, e, \dot{c})$ is the sectional curvature of M^3 and $(\nabla f)^\perp$ means the projection of ∇f onto e.

We choose $f = g^k$ for some number k which will be determined later, and use the fact that g satisfies equation (1) on Σ, we have

$$k(k-1)\int_0^l \varphi^2 g^{k-2}|\nabla g|^2 - \int_0^l \varphi^2 \frac{d^2 g^k}{dt^2} + \int_0^l g^k \dot{\varphi}^2 - \int_0^l \varphi^2 g^{-k}|(\nabla g^k)^\perp|^2$$
$$\geq \int_o^l \varphi^2 g^k \{k|h|^2 + kRic(\nu) + R(e, \dot{c}, e, \dot{c}) - (h(\dot{c}, \dot{c}))^2\}.$$

Replace $\varphi^2 g^k$ by φ^2 and integrate by parts, then one can easily see that the last inequality implies

$$(\frac{1}{4}k^2 - k)\int_0^l \varphi^2 |(\ln g)'|^2 + k\int_0^l \varphi\varphi'(\ln g)' + \int_0^l \dot{\varphi}^2$$
$$\geq \int_0^l \varphi^2\{k|h|^2 - (h(\dot{c}, \dot{c}))^2 + kRic(\nu) + R(e, \dot{c}, e, \dot{c})\}$$

If we choose $0 < k < 4$, then by completing the square of the left-hand-side of the last inequality, we get

$$\frac{4}{4-k}\int_0^l \dot{\varphi}^2 \geq \int_0^l \varphi^2\{k|h|^2 - (h(\dot{c}, \dot{c}))^2 + kRic(\nu) + R(e, \dot{c}, e, \dot{c})\}$$

In order to find the smallest possible upper bound for l from the previous inequality, it is not difficult to see that one should choose $k = \frac{7}{4}$. Therefore, the last inequality becomes

$$\frac{16}{9}\int_0^l \dot{\varphi}^2 \geq \int_0^l \varphi^2\{\frac{3}{4}Ric(\nu) + \frac{1}{2}S\}$$

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where S is the scalar curvature of the ambient manifold.

Since $Ric \geq 2$, we obtain

$$\frac{32}{81} \int_0^l \dot{\varphi}^2 \geq \int_0^l \varphi^2$$

Let $\phi = \sin \frac{\pi}{l} s$, it follows that

$$l \leq \frac{4\sqrt{2}}{9} \pi.$$

<div align="right">q.e.d.</div>

Lemma 3 *Let $M \subset int(N)$ be two compact orientable three-manifolds with nonempty boundary. If $\pi_2(M) \to \pi_2(N)$ is trivial, then $\pi_1(M)$ is torsion free.*

This is lemma 3.5 in [Zh], see there for a proof.

<u>*Proof of Theorem 1*</u> By the well-known theorem of Hamilton ([Ha]), we only need to prove that $\pi_1(M,p) = \{e\}$ (We remark that in higher dimensions, M is not necessarily simply connected under the Ricci curvature condition.) We can also assume that M is oriented. In fact, if M is not orientable, we can use the same argument to the orientable double cover to conclude that M has to be RP^3, which is orientable, thus a contradiction. In what follows, we will assume all geodesic balls correspond to regular values of distance function (or its smoothing), thus are all manifolds with boundary. The main idea of the proof is to produce a minimal surface of sufficient big radius if the manifold is not simply connected. We will first prove the following,

<u>*Claim*</u>: For any r with $\tau < r < \text{diam} - 2\tau$, $\pi_1(B_p(r))$ is torsion free.

In fact, take R: $r < R < \text{diam} - \tau$. consider the inclusion $i : B_p(r) \to B_p(R)$, we only need to show that $i_*(\pi_2(B_p(r))$ is trivial. The claim then follows from Lemma 3.

We proceed by contradiction. Assume $i_*(\pi_2)$ is not trivial, by the sphere theorem in 3-dimensional topology, there is an embedded S^2 in $B_p(r)$ which is not null-homotopic in $B_p(R)$. This leads to three situations, each will lead to a contradiction, as we now discuss.

Case 1. S^2 does not seperate $B_p(R)$.

From Lemma 3.8 in [He], $B_p(R) = M_1 \sharp M_2$, where M_1 is a S^2 bundle over S^1. In particular, $\pi_1(M_1) = Z$. But $M = M_1 \sharp M_3$ for some M_3. But

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$M_3 \supset B_q(\tau)$, so Lemma 1 implies that $\pi_1(M_3) \neq \{e\}$. It follows from Van Kampen's theorem that $\pi_1(M)$ is infinite, contradicting Myers theorem.

Case 2. S^2 seperates $B_p(R)$ into two components, both of which have non-empty intersection with $\partial B_p(R)$.

Denote $B_p(R) \setminus S^2 = M_1 \cup M_2$, take $x_i \in M_i \cap \partial B_p(R)$. Let γ be a curve connecting x_1 to x_2 in $B_p(R)$, then γ intersects S^2 at a single point. Take another curve σ from x_2 to x_1 in $M \setminus B_p(R)$, then $\sigma \circ \gamma$ is a closed loop in M, intersecting S^2 at a single point. Such a loop has infinite order in $\pi_1(M)$, again contradicting Myers theorem.

Case 3. S^2 seperates $B_p(R)$ into two connected components, $B_p(R) \setminus S^2 = M_1 \cup M_2$, of which $M_1 \cap \partial B_p(R) \neq \emptyset$ and $M_2 \cap \partial B_p(R) = \emptyset$ (i.e., $\partial(M_2) = S^2$.)

It follows $B_p(R) = M_1 \sharp M_2$. If $\pi_1(M_2) = \{e\}$, then M_2 is a homotopy three ball, thus the embedded S^2 is trivial in $\pi_2(B_p(R))$, contradicting our choice of S^2. Therefore $\pi_1(M_2) \neq \{e\}$. But then $M = M_2 \sharp M_3$ for some M_3. If M_3 is not simply connected, Van Kampen's theorem will again imply $\pi_1(M)$ is infinite, contradicting Myer's theorem. Thus $\pi_1(M_3) = \{e\}$. But $M_3 \supset B_q(\tau)$, therefore any loop in $B_q(\tau)$ is null-homotopic in M_3. This by Lemma 1 implies $\pi_1(M, q)$ is trivial, contradicting the assumption. This completes the proof of the claim.

Consider now the inclusions

$$B_p(\tau) \xrightarrow{i} B_p(R) \xrightarrow{j} B_p(R + \delta)$$

where $R = \pi - \epsilon - 2\tau$, and $\delta < \tau$. Then

$$\pi_1(B_p(\tau)) \xrightarrow{i_*} \pi_1(B_p(R)) \xrightarrow{j_*} \pi_1(B_p(R + \delta)).$$

By the claim, the middle group is torsion free, thus $i_*(\pi_1(B_p(\tau))$ is either trivial or has infinite order. The first case is not possible since it will imply $\pi_1(M)$ is trivial. Thus $i_*(\pi_1(B_p(\tau))$ is infinite.

Take a geodesic loop σ at p representing an element of $\pi_1(M, p)$ such that $i_*(\sigma)$ has infinite order. Since $\pi_1(M, p)$ is finite, there is an $m > 0$ such that $m \cdot \sigma$ is trivial in $\pi_1(M, p)$. Purturbe this curve to have it embedded (still denoted by $m \cdot \sigma$). We then solve the Platau problem in M with $m \cdot \sigma$ as boundary to get an area minimizing surface Σ (a disc). Since $i_*(\sigma)$ has infinite order in $\pi_1(B_p(R))$, $m \cdot \sigma$ is not trivial in $B_p(R)$, thus $\Sigma \cap (M \setminus B_p(R)) \neq \emptyset$, that is, there is $x \in \Sigma$ such that $\text{dist}(x, p) > R = \pi - \epsilon - 2\tau$. This is not possible by Lemma 2. Direct computation from the above inequality and that of Lemma 2 shows that $\epsilon \approx 0.47\pi$ is enough. q.e.d.

References

[An]   M. Anderson, *Metrics of Positive Ricci curvature with large diameter*, Manu. Math., **68** (1990), 405-415.

[Ch]   S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math Z., **143** (1975), 289-297.

[Cr]   C. Croke, *An eigenvalue pinching theorem*, Invent. Math., **68** (1982), 253-256.

[Es]   J. Eschenburg, *Diameter, volume and topology for positive Ricci curvature*, J. Diff. Geom., **33** (1991), 743-747 .

[FS]   D. Fischer-Colbrie, R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature*, Comm. Pure Appl. Math, **33** (1980), 199-211.

[GP]   K. Grove, P. Petersen, *On the excess of metric spaces and manifolds*, Preprint.

[GS]   K. Grove, K. Shiohama, *A generalized sphere theorem*, Ann. of Math., **106** (1977), 201-211.

[Ha]   R. Hamilton, *Three manifolds with positive Ricci curvature*, J. Diff. Geom.,

[He]   Hemple, *3-manifolds*, Ann. of Math Studies, **86**, Princeton Univ. Press, 1976.

[Li]   A. Lichnerowicz, *Geometrie des groupes des transformations*, Dunod, Paris (1958).

[LT]   P. Li, A. Treibergs, *Pinching theorem for the first eigenvalue on positively curved four manifolds*, Invent. math., **66** (1982), 35-38.

[LZ]   P. Li, J.-Q. Zhong, *Pinching theorem for the first eigenvalue on positively curved manifolds*, Invent. math., **65** (1981), 221-225.

[Ob]   M. Obata, *Certain conidtions for a Riemannian manfiold to be isometric with a sphere*, J. Math. Soc. Japan, **14** (1962), 333-340.

[Ot]   Y. Otsu, *On manifolds of positive Ricci curvature with large diameter*, Math. Z., **206** (1991), 255-264.

[Pe]   G. Perelman, *The diameter sphere theorem*, Preprint.

[Pi]   M. Pinsky, *A topological version of Obata's sphere Theorem*, J. Diff. Geom., **14** (1979), 369-376.

[PZ]   P. Petersen, S. h. Zhu, *An excess sphere theorem*, Ann. Scient. Ec. Norm. Sup., **26** (1993), 175-188.

[Sh]   K. Shiohama, *A sphere theorem for manifolds of positive Ricci curvature*, Trans. AMS., **275** (1983), 811-819.

[SY1]   R. Scheon, S.-T. Yau, *Complete three dimensional manifolds with positive Ricci curvature and scalar curvature*, Ann. Math. Studies, **102** (1982), 209-228.

[SY2]   ——————————, *The existence of a black hole due to condensation of matter*, Comm. Math. Phys. **90** (1983), 575-579.

[To]   V. Toponogov, *Riemannian spaces having their curvature bounded below by a positive number*, AMS Translations, **37** (1964, 291-336.

[Wu]   J.-Y. Wu, *Convergence of Riemannian 3-manifolds under Ricci curvature bounds* , Preprint.

[Zh]   S. h. Zhu, *A finiteness theorem for Ricci curvature in dimension three*, J. Diff. Geom., **37** (1993), 711-727.